

# Energy–Momentum Tensor for the Electromagnetic Field in a Dielectric

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The total momentum of a thermodynamically closed system is unique, as is the total energy. Nevertheless, there is continuing confusion concerning the correct form of the momentum and the energy–momentum tensor for an electromagnetic field interacting with a linear dielectric medium. Here we investigate the energy and momentum in a closed system composed of a propagating electromagnetic field and a negligibly reflecting dielectric. The Gordon momentum is easily identified as the total momentum by the fact that it is, by virtue of being invariant in time, conserved. We construct continuity equations for the energy and the Gordon momentum and use the continuity equations to construct an array that has the properties of a traceless, diagonally symmetric energy–momentum tensor. Then the century-old Abraham–Minkowski momentum controversy can be viewed as a consequence of attempting to construct an energy–momentum tensor from continuity equations that contain densities that correspond to nonconserved quantities.

## I. INTRODUCTION

The energy–momentum tensor is a concise way to represent the conservation properties of the flow field. For most types of simple flows, the energy–momentum tensor is well-defined, with the notable exception of the electromagnetic field in a linear dielectric material. The Abraham–Minkowski controversy [1–9] for the momentum of electromagnetic fields in a dielectric began with the derivation of the energy–momentum four-tensor by Minkowski [10]. Noting that the Minkowski tensor is not diagonally symmetric and therefore contains problems with conservation of angular momentum, Abraham [11] proposed an energy–momentum tensor that was symmetric, but at the expense of a new phenomenological force. In order to address this constraint and additional issues, Einstein and Laub [7], Nelson [8], and others proposed variants of the energy–momentum tensor.

The crux of the Abraham–Minkowski controversy is whether the electromagnetic momentum density in a dielectric is of the Minkowski form

$$\mathbf{g}_M = \frac{1}{c}(\mathbf{D} \times \mathbf{B}) \quad (1.1)$$

or the Abraham form

$$\mathbf{g}_A = \frac{1}{c}(\mathbf{E} \times \mathbf{H}). \quad (1.2)$$

Experimental efforts to resolve the theoretical impasse in one direction or the other have not been definitive. While some experiments favor the Abraham formula, other experiments support Minkowski’s version. Brevick’s [12] analysis of experiments performed by Jones and Richards [13], Ashkin and Dziedzic [14], and others showed that the allocation of momentum between the field and material was the determining factor in whether a particular experimental result was described by the Abraham or Minkowski form of electromagnetic momentum. Following Brevick [12], the formula for the field momentum has been shown repeatedly to be essentially arbitrary such that any of the formulas for the field momentum can be

combined with an appropriate momentum for the material to produce the same total momentum [15].

In 1973, Gordon [1] constructed the total momentum from a microscopic model in which the electromagnetic field component of the total momentum is said to be the Abraham momentum and the dielectric is treated as a dilute collection of electric dipoles with center-of-mass motion in the direction of propagation of the field. Gordon [1] discusses the empirical and experimental validation of the total momentum density

$$\mathbf{g}_G = \frac{n}{c}(\mathbf{E} \times \mathbf{B}) \quad (1.3)$$

and shows that the density  $\mathbf{g}_G$ , integrated over a volume containing the entire field, is invariant in time. Mikura [16] subsequently obtained the total momentum using a more general material model that included polarizability, magnetizability, electrostriction, and other material considerations. Since then, the total momentum has been constructed using a variety of field and material momenta [15].

The historical development of the electromagnetic energy–momentum tensor for the field in a dielectric has been intertwined with efforts to derive the correct electromagnetic momentum density. The original Minkowski and Abraham tensors, like the Minkowski and Abraham momenta, are now regarded as being associated with the field, alone, requiring a material component to complete the total energy–momentum tensor [15]. Although a number of composite energy–momentum tensors have been constructed [15], the tensor forms that have been constructed from field and material constituents fail to satisfy the constraints imposed on the symmetry, trace, or four-divergence of a total energy–momentum tensor.

The basic tenant of classical continuum electrodynamics is that the electrodynamic properties of a linear material can be characterized by macroscopic parameters. In this long wavelength limit, the material is modeled as a simple linear dielectric with a linear refractive index  $n$ . In reflection, the electromagnetic field exerts radiation pressure on the dielectric. Neither the motion of an

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unrestrained dielectric nor the forces of restraint on a stationary dielectric are coupled to the macroscopic Maxwell equations of motion for the electromagnetic field. For that reason, we limit our consideration to the case of negligible reflection in which the consequences of radiation pressure vanish. The total electromagnetic momentum for the closed system consisting of the macroscopic electromagnetic field and a negligibly reflecting linear dielectric is unique. Here, we identify the Gordon momentum [1] as the total momentum by the fact that it is invariant in time and therefore conserved in an isolated system. The total energy is shown to be similarly conserved. In order to satisfy the constraints imposed on an energy-momentum tensor, we perform a simple change of variables and derive continuity equations for the total energy and the total momentum. The continuity equations are then used to construct an array that has the properties of a traceless, diagonally symmetric energy-momentum tensor, but in a coordinate system with time-like coordinate  $ct/n$ .

## II. ENERGY-MOMENTUM TENSOR OF NONINTERACTING PARTICLES

In the continuum limit, the density of any property of identical noninteracting particles can be treated as the number density multiplied by the amount of the property that can be attributed to each particle. The continuity equation corresponding to a specific property, such as mass, charge, or energy, is then obtained by substitution of the specific property density for a placeholder number density. For an infinitesimal element of volume in an inviscid sourceless flow, the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \quad (2.1)$$

is derived by applying the divergence theorem to a Taylor series expansion of the property density field  $\rho$  and the vector velocity field  $\mathbf{u} = (u^x, u^y, u^z)$  of the flow [17]. The continuity equation reflects the conservation of a continuous scalar property in a flow in terms of the equality of the net rate of flux out of the volume and the time rate of change of the property density  $\rho$  inside the volume. Depending on the context, the three-vector

$$\mathbf{g} = \rho \mathbf{u} \quad (2.2)$$

is known as the momentum density, the flux, or the current density of the property. Specifically,  $\mathbf{g}$  corresponds to the linear momentum density if  $\rho$  is a mass density and corresponds to the charge current density if  $\rho$  is the electric charge density.

Some conserved properties, such as momentum, are vectors. For a flow, the density of a conserved vector property can be represented as  $\boldsymbol{\rho} = (\rho^x, \rho^y, \rho^z)$ . Applying the scalar formalism to the three orthogonal components of the property density vector yields scalar continuity

equations

$$\frac{\partial \rho^x}{\partial t} + \nabla \cdot \rho^x \mathbf{u} = 0 \quad (2.3a)$$

$$\frac{\partial \rho^y}{\partial t} + \nabla \cdot \rho^y \mathbf{u} = 0 \quad (2.3b)$$

$$\frac{\partial \rho^z}{\partial t} + \nabla \cdot \rho^z \mathbf{u} = 0. \quad (2.3c)$$

At this point, we adopt a four-dimensional notation where repeated indices are summed. We take Roman indices to run from 1 to 3 and we identify the coordinates  $x^i$  with the Cartesian coordinates, such that  $x^1 = x$ ,  $x^2 = y$ , and  $x^3 = z$ . Greek indices run from 0 to 3 and  $x^0$  is identified with the time-like coordinate  $ct$ . The Minkowski space-time metric is  $diag(-1, 1, 1, 1)$ . Finally, partial differentiation with respect to the indexed coordinates is represented by  $\partial_\alpha = \partial/(\partial x^\alpha)$ .

The four continuity equations (2.1) and (2.3) can be concisely represented by

$$\partial_\beta G^{\alpha\beta} = 0, \quad (2.4)$$

where

$$G^{\alpha\beta} = \begin{bmatrix} c^2 \rho & cu^x \rho & cu^y \rho & cu^z \rho \\ c\rho^x & u^x \rho^x & u^y \rho^x & u^z \rho^x \\ c\rho^y & u^x \rho^y & u^y \rho^y & u^z \rho^y \\ c\rho^z & u^x \rho^z & u^y \rho^z & u^z \rho^z \end{bmatrix}. \quad (2.5)$$

In order for this representation to be unique, we must specify the relation between the continuity equations, that is, the relation between elements in different rows of  $G^{\alpha\beta}$  in Eq. (2.5) [18]. For a closed system, conservation of angular momentum requires  $G^{\alpha\beta}$  to be symmetric [18]

$$G^{\alpha\beta} = G^{\beta\alpha}. \quad (2.6)$$

Therefore, the vector property  $\boldsymbol{\rho}$  must represent the flux of the conserved scalar quantity  $\rho$  such that  $\boldsymbol{\rho} = \rho \mathbf{u}$ . Conversely, the flux  $\rho \mathbf{u}$  of a scalar property of the particles is a conserved vector property of the flow. The covariant form of the continuity equation is the four-divergence, Eq. (2.4), of the energy-momentum four-tensor

$$G^{\alpha\beta} = \begin{bmatrix} c^2 \rho & c\rho u^x & c\rho u^y & c\rho u^z \\ c\rho u^x & \rho u^x u^x & \rho u^x u^y & \rho u^x u^z \\ c\rho u^y & \rho u^y u^x & \rho u^y u^y & \rho u^y u^z \\ c\rho u^z & \rho u^z u^x & \rho u^z u^y & \rho u^z u^z \end{bmatrix}. \quad (2.7)$$

The continuity equation (2.4) with the four-tensor (2.7) is valid for any conserved extensive quantity in a simple flow, not just for the mass of a fluid.

The energy-momentum tensor (2.7) has some essential properties. First, the four-divergence of each row vector

$$\partial_\beta G^{\alpha\beta} = 0 \quad (2.8)$$

is a continuity law corresponding to the conservation of the property represented by the property density in the first element of that row. Second, the tensor is diagonally symmetric

$$G^{\alpha\beta} = G^{\beta\alpha} \quad (2.9)$$

corresponding to the absence of unbalanced shear forces and conservation of angular momentum in a closed system [18]. Third, as a consequence of diagonal symmetry,

$$\partial_\alpha G^{\alpha\beta} = 0, \quad (2.10)$$

the four-divergence of each column vector is a continuity law corresponding to the conservation of the property represented by the property density in the first element of that column. It should also be noted that the simple energy–momentum tensor is based on the properties of an unimpeded flow. If the flow is redirected by impact with a macroscopic object then one is obligated to include the equations of motion of the object or the forces of restraint.

### III. THE ABRAHAM AND MINKOWSKI ENERGY–MOMENTUM TENSORS

The Abraham and Minkowski energy–momentum tensors are examples of a number of different tensors that have been proposed for the electromagnetic field in a dielectric [15]. The Minkowski tensor can be constructed from the continuity equations for energy flux and momentum flux by the same procedure that was used to construct the array (2.5). The Minkowski tensor is not diagonally symmetric and therefore would admit violations of angular momentum conservation if it were to be considered the total energy–momentum tensor of a closed system. Instead, the Minkowski tensor is considered to be a representation of the energy and momentum of the field-only component of a system. In this section, we outline the construction of the Minkowski tensor and describe how the procedure is modified to obtain the Abraham tensor. Neither the Minkowski tensor, nor the Abraham tensor, satisfy the requirements of a total energy–momentum tensor. The Minkowski tensor is not symmetric while the Abraham tensor contains a phenomenological volume force.

The macroscopic Maxwell equations of continuum electrodynamics are the basis for deriving continuity equations for electric and magnetic fields in a dielectric. For a dielectric with no free charges in a regime of negligible absorption and dispersion, the Maxwell equations may be written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial(ct)} \quad (3.1a)$$

$$\nabla \times \mathbf{B} = \frac{\partial n^2 \mathbf{E}}{\partial(ct)} \quad (3.1b)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.1c)$$

$$\nabla \cdot \frac{n^2}{c^2} \mathbf{E} = 0 \quad (3.1d)$$

in Heaviside–Lorentz units. The electric and magnetic fields can be defined in terms of the vector potential  $\mathbf{A}$  as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial(ct)} \quad (3.2a)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (3.2b)$$

for transverse fields in the Coulomb gauge.

The macroscopic Maxwell equations are the axioms of classical continuum electrodynamics. Poynting's theorem,

$$\frac{\partial}{\partial(ct)} \left[ \frac{1}{2} (n^2 \mathbf{E}^2 + \mathbf{B}^2) \right] + \nabla \cdot (\mathbf{E} \times \mathbf{B}) = 0, \quad (3.3)$$

can be derived by multiplying the Faraday law (3.1a) by  $\mathbf{B}$  and adding it to the Maxwell–Ampère law (3.1b) multiplied by  $\mathbf{E}$ . Poynting's theorem can also be derived by substituting the Maxwell equations into the temporal derivative of the energy density

$$\rho_e = (1/2)(n^2 \mathbf{E}^2 + \mathbf{B}^2) \quad (3.4)$$

using a vector triple-product identity. Poynting's theorem is a continuity equation for the Poynting energy-flux vector

$$\mathbf{S}_P = c(\mathbf{E} \times \mathbf{B}) = (s_P^1, s_P^2, s_P^3). \quad (3.5)$$

The theorem

$$\begin{aligned} \frac{\partial}{\partial(ct)} (n^2 \mathbf{E} \times \mathbf{B}) &= -\mathbf{B} \times (\nabla \times \mathbf{B}) + \mathbf{B}(\nabla \cdot \mathbf{B}) \\ &\quad - n^2 \mathbf{E} \times (\nabla \times \mathbf{E}) + \mathbf{E}(\nabla \cdot n^2 \mathbf{E}) \end{aligned} \quad (3.6)$$

is derived, in a manner similar to Poynting's theorem, by substituting Maxwell's equations into the temporal derivative of the Minkowski momentum density,  $\mathbf{g}_M = (n^2/c)(\mathbf{E} \times \mathbf{B})$ . The right-hand side of Eq. (3.6) can be recast, approximately, as the negative of the divergence of the Maxwell stress tensor [19] with components

$$W^{ij} = \left[ -n^2 E_i E_j - B_i B_j + \frac{1}{2} (n^2 \mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}) \delta_{ij} \right], \quad (3.7)$$

where terms involving the gradient of  $n^2$  have been neglected. Then the temporal derivative of the Minkowski momentum density, Eq. (3.6), can be expressed using the vector divergence operator,  $\nabla \cdot$ , as

$$\frac{\partial}{\partial(ct)} (n^2 \mathbf{E} \times \mathbf{B}) + \nabla \cdot \mathbf{W} = 0. \quad (3.8)$$

The array

$$T_M^{\alpha\beta} = \begin{bmatrix} \rho_e & s_P^1/c & s_P^2/c & s_P^3/c \\ cg_M^1 & W^{11} & W^{12} & W^{13} \\ cg_M^2 & W^{21} & W^{22} & W^{23} \\ cg_M^3 & W^{31} & W^{32} & W^{33} \end{bmatrix}, \quad (3.9)$$

known as the Minkowski energy–momentum tensor, is constructed from continuity equations (3.3) and (3.8). Using the summation convention, we can write Poynting’s theorem in Eq. (3.3) as

$$\partial_0 T_M^{00} + \partial_j T_M^{0j} = 0 \quad (3.10)$$

for continuity of the energy flux and Eq. (3.8) as

$$\partial_0 T_M^{i0} + \partial_j T_M^{ij} = 0 \quad (3.11)$$

for continuity of the momentum flux. Then, each row of  $T_M$  corresponds to a four-divergence

$$\partial_\beta T_M^{\alpha\beta} = 0. \quad (3.12)$$

Because the array  $T_M^{\alpha\beta}$  in Eq. (3.9) is not diagonally symmetric, Abraham proposed the energy–momentum tensor

$$T_A^{\alpha\beta} = \begin{bmatrix} \rho_e & s_P^1/c & s_P^2/c & s_P^3/c \\ cg_A^1 & W^{11} & W^{12} & W^{13} \\ cg_A^2 & W^{21} & W^{22} & W^{23} \\ cg_A^3 & W^{31} & W^{32} & W^{33} \end{bmatrix}, \quad (3.13)$$

where  $\mathbf{g}_A$  is given by Eq. (1.2). The four divergence of this tensor is

$$\partial_\beta T_A^{\alpha\beta} = -f^\alpha, \quad (3.14)$$

where  $f^\alpha$  is the Abraham force. The Cartesian components of the Abraham force

$$\mathbf{f} = \frac{\partial}{\partial(ct)} ((n^2 - 1)\mathbf{E} \times \mathbf{B}) \quad (3.15)$$

are obtained by substituting the Abraham momentum density into the continuity equation (3.8) and the time-like coordinate component is  $f^0 = 0$ .

It has been widely reported in the literature that neither  $T_M$  nor  $T_A$  is the total energy–momentum tensor [15]. Instead, they are considered to be two of many arbitrary forms of the electromagnetic part,  $T_{\text{fld}}$ , of a total energy momentum tensor

$$T = T_{\text{fld}} + T_{\text{matl}} \quad (3.16)$$

composed of energy–momentum tensors for the field and material subspaces, however those subspaces are defined.

#### IV. TOTAL MOMENTUM

A well-defined quantity for the physical momentum is not derivable solely from the macroscopic Maxwell equations without additional conditions. However, the total

momentum of an isolated system must be constant in time. In this section, we identify the unique total momentum using this constraint. We consider the case of a quasi-monochromatic electromagnetic field, in the plane-wave limit, entering a linear medium from vacuum at normal incidence. The medium is taken to be a simple linear dielectric in the regime of negligible dispersion and absorption. A dielectric slab is partially reflecting and the material acquires a kinematic momentum that is twice the momentum of the reflected field, but in the direction of the incident field. In the absence of reflection, there is no momentum given to the dielectric slab and it remains stationary. We adopt this case of a stationary dielectric in which reflections can be neglected by assuming that an antireflection coating has been applied to the dielectric or that the refractive index of the dielectric is only slightly greater than unity.

Gordon [1] used a microscopic model of the dielectric as a vapor of weakly polarizable atoms and derived the material momentum as the continuum average of the mechanical momentum of the atoms. Assuming a rarefied vapor of atoms in order for reflections to be negligible, Gordon obtained the momentum

$$\mathbf{G}_G = \int_V dv \mathbf{g}_G = \int_V dv \frac{n}{c} (\mathbf{E} \times \mathbf{B}) \quad (4.1)$$

by adding the material momentum to the Abraham momentum. In continuum electrodynamics, the electrodynamic properties of a material are characterized only by a macroscopic refractive index. Therefore, the microscopic origin of the Gordon momentum (4.1) is of no consequence in the formalism of continuum electrodynamics. Following Gordon, we will demonstrate that the momentum (4.1) is invariant in time and, because it is conserved, can be identified as the total momentum. Because the Gordon momentum depends on the material only through the refractive index, it is the unique total momentum for all cases in which the medium behaves, to a good approximation, as a linear dielectric with refractive index  $n$ .

Propagation of a field in a linear medium is governed by the wave equation,

$$\nabla^2 \mathbf{A} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \quad (4.2)$$

written in terms of the vector potential  $\mathbf{A}$ ,  $\mathbf{B} = \nabla \times \mathbf{A}$ . For quasi-monochromatic plane waves, it is convenient to write the vector potential in terms of a slowly varying envelope function,  $A(z, t)$ , a rapidly varying carrier, and a unit vector,  $\mathbf{e}_k$ , in the direction of propagation as  $\mathbf{A} = A(z, t) e^{-i(\omega_0 t - kz)} \mathbf{e}_k$ . For a plane-wave entering a dielectric at normal incidence, reflections are negligible if  $\delta n = n - 1$  is small. In this limit, there is no momentum given to the bulk material, which remains stationary, and we can apply the Fresnel relation

$$\frac{A_t}{A_i} = \frac{2}{n+1} = \left(1 + \frac{\delta n}{2}\right)^{-1}, \quad (4.3)$$

where  $A_i$  is the incident amplitude and  $A_t$  is the transmitted amplitude. Comparing Eq. (4.3) with a series expansion of  $1/\sqrt{n}$  in the limit of small  $\delta n$ , we find that the vector potential amplitude inside the dielectric,  $A_t$ , is reduced by a factor of  $\sqrt{n}$  from the incident amplitude  $A_i$ .

For plane cw waves, and approximately for slowly varying waves, the relation between the amplitudes of the fields simplifies to

$$|\mathbf{B}| = n|\mathbf{E}| = \frac{n\omega}{c}|\mathbf{A}|. \quad (4.4)$$

Then, the electromagnetic energy density,  $\rho_e = (1/2)(n^2\mathbf{E}^2 + \mathbf{B}^2)$ , can be written as

$$\rho_e = \frac{n^2\omega^2}{c^2}|\mathbf{A}|^2. \quad (4.5)$$

Applying the Fresnel amplitudes to relate the fields inside and outside the medium,  $A_t = A_i/\sqrt{n}$ , we find that the energy density inside the material is a factor of  $n$  larger than the energy density of the same field in the vacuum. The region occupied by the field in the material is compressed spatially by a factor of  $n$  due to the reduced speed of light within the medium, such that the total energy

$$U = \int_V dv \rho_e = \int_V dv \frac{n^2\omega^2}{c^2}|\mathbf{A}|^2 \quad (4.6)$$

is conserved. Numerical solutions of the wave equation for a field entering a linear material through a gradient-index anti-reflection coating indicate that the field in the material is a factor of  $\sqrt{n}$  smaller and a factor of  $n$  narrower than the field in the vacuum, independent of the magnitude of  $n$ , as long as reflections are suppressed [20].

Having demonstrated the conservation properties of the electromagnetic energy, we demonstrate the conservation properties of the Gordon electromagnetic momentum by a similar procedure. The Gordon momentum [1] is obtained by integrating the momentum density (1.3) over all three-dimensional space. Comparing the Gordon momentum, expressed in terms of the envelope functions

$$\mathbf{G}_G = \int_V dv \frac{n}{c}(\mathbf{E} \times \mathbf{B}) = \int_V dv \frac{n^2\omega^2}{c^3}|\mathbf{A}|^2 \mathbf{e}_k, \quad (4.7)$$

with the total energy, Eq. (4.6), we see that conservation of total energy implies conservation of the Gordon momentum. The total momentum of a closed system is unique and the Gordon form of total momentum is conserved. Therefore, the Gordon momentum can be identified as the total momentum of the thermodynamically closed system. By uniqueness, neither the Minkowski momentum nor the Abraham momentum is conserved.

We point out that, in the macroscopic limit in which the dielectric is described by a refractive index  $n$ , the question of what portion of the energy given by Eq. (4.6) resides in the field or dielectric is improperly posed. Comparing Eqs. (4.6) and (4.7), we see that the same holds true for the apportionment of the momentum into field and dielectric components.

## V. TOTAL ENERGY–MOMENTUM TENSOR

In the previous section, we identified the unique total momentum for the system of an electromagnetic field in a dielectric for the case of negligible reflections. In this section, we construct the corresponding total energy–momentum tensor from continuity equations for the energy and momentum.

A continuity equation is a differential form of a conservation law applied to an element of volume in a continuous flow. In Sec. II we showed that the energy–momentum tensor for dust is constructed from continuity equations in which the differential operators act on the densities of conserved quantities. The operand of the time derivative in the continuity equation (3.8) is the Minkowski momentum density. Because the volume integral of the Minkowski momentum density is not a conserved vector quantity, we do not consider Eq. (3.8) to be a suitable continuity equation with which to construct an energy–momentum tensor. Instead, we write Eq. (3.8) as

$$\frac{n}{c} \frac{\partial}{\partial t} c\mathbf{g}_G + \nabla \cdot \mathbf{W} = 0, \quad (5.1)$$

in terms of the Gordon momentum density  $\mathbf{g}_G = (n/c)(\mathbf{E} \times \mathbf{B})$  whose volume integral is a conserved vector quantity. Equation (5.1) provides three continuity equations for our energy–momentum tensor. The additional continuity equation is obtained by writing Poynting's theorem in (3.3) as

$$\frac{n}{c} \frac{\partial \rho_e}{\partial t} + \nabla \cdot [n(\mathbf{E} \times \mathbf{B})] = \frac{\nabla n}{n} \cdot n(\mathbf{E} \times \mathbf{B}) \quad (5.2)$$

using the densities of conserved quantities. We are considering the case of a closed system in which there are no reflections. As this is assumed to be accomplished by a gradient-index antireflection coating, we can drop the term containing  $\nabla n$  and write Poynting's theorem in Eq. (5.2) as

$$\frac{n}{c} \frac{\partial \rho_e}{\partial t} + \nabla \cdot [n(\mathbf{E} \times \mathbf{B})] = 0. \quad (5.3)$$

In Maxwell's equations, expressed in terms of 3-vectors, time is not a coordinate. We make a simple change of time variable to  $\tau = t/n$  and write the continuity theorems (5.1) and (5.3) as

$$\frac{1}{c} \frac{\partial}{\partial \tau} c\mathbf{g}_G + \nabla \cdot \mathbf{W} = 0 \quad (5.4a)$$

$$\frac{1}{c} \frac{\partial \rho_e}{\partial \tau} + \nabla \cdot c\mathbf{g}_G = 0, \quad (5.4b)$$

where the Gordon momentum density is given in Eq. (1.3). However when writing Maxwell's equations as tensor equations, time is one of the four space–time coordinates and we define the time-like coordinate

$$\bar{x}^0 = c\tau = \frac{ct}{n}. \quad (5.5)$$

Then the four scalar continuity equations, Eqs. (5.4a) and (5.4b), can be written concisely as a single equation, as in Section 2. We define an operator

$$\bar{\partial}_\alpha = \left( \frac{\partial}{\partial \bar{x}^0}, \partial_x, \partial_y, \partial_z \right) \quad (5.6)$$

and an array

$$T^{\alpha\beta} = \begin{bmatrix} \rho_e & cg_G^1 & cg_G^2 & cg_G^3 \\ cg_G^1 & W^{11} & W^{12} & W^{13} \\ cg_G^2 & W^{21} & W^{22} & W^{23} \\ cg_G^3 & W^{31} & W^{32} & W^{33} \end{bmatrix}, \quad (5.7)$$

such that

$$\bar{\partial}_\beta T^{\alpha\beta} = 0. \quad (5.8)$$

The array (5.7) has a number of notable properties. The array is diagonally symmetric

$$T^{\alpha\beta} = T^{\beta\alpha} \quad (5.9)$$

and has a vanishing trace

$$T^\alpha_\alpha = 0. \quad (5.10)$$

The operator (5.6) applied to the rows of (5.7) generates a continuity law for a demonstrably conserved electromagnetic energy or momentum property. The operator (5.6) applied to the columns

$$\bar{\partial}_\alpha T^{\alpha\beta} = 0 \quad (5.11)$$

generates the same continuity equations as a consequence of symmetry. These are the properties that we associate with an energy–momentum tensor.

## VI. CONCLUDING REMARKS

For many years, the Abraham–Minkowski controversy has been resolved by postulating a total energy–momentum tensor that is comprised of separate field and matter tensors, an approach that involves assumptions about the behavior of matter in the presence of an electromagnetic field. In continuum electrodynamics the interaction of the field and matter is described in terms of a single macroscopic parameter, the refractive index  $n$ . We showed that the total momentum of a thermodynamically closed system consisting of a quasimonochromatic field and negligibly reflecting linear dielectric in the continuum limit is the Gordon momentum. We derived continuity equations from the Maxwell equations and used a time variable transformation to write the continuity equations in terms of densities of conserved energy and Gordon momentum quantities. When written in four-dimensional tensor form with time-like coordinate  $ct/n$ , the continuity equations are obtained from the four-divergence of a traceless, diagonally symmetric energy–momentum tensor.

In summary, we constructed the energy–momentum tensor in Eq. (5.7) from continuity equations that were derived from the macroscopic Maxwell’s equations. It is interesting to note that a time coordinate transformation was required in order to write the continuity equations as the four-divergence of the symmetric energy–momentum tensor.

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